

SUPERPOSITION OF FIELDS OF TWO REISSNER - NORDSTRÖM SOURCES

G.A. Alekseev

Steklov Mathematical Institute, Gubkina 8, Moscow 119991, Moscow, Russia
G.A.Alekseev@mi.ras.ru

V.A. Belinski

*INFN, Rome University "La Sapienza", 00185 Rome, Italy,
ICRANet, Piazzale della Repubblica, 10, 65122 Pescara, Italy
and IHES, F-91440 Bures-sur-Yvette, France*
belinski@icra.it

In this paper we present a 5-parametric family of static asymptotically flat solutions for the superposed gravitational and electromagnetic fields of two Reissner-Nordström sources with arbitrary parameters — masses, charges and separating distance. A procedure for solving of the linear singular integral equation form of the electrovacuum Einstein - Maxwell equations for stationary axisymmetric fields is described in detail. The 4-parametric family of equilibrium configurations of two Reissner-Nordström sources (one of which should be a black hole and another one — a naked singularity) presented in our recent paper¹ arises after a restriction of the parameters of the 5-parametric solution presented here by the equilibrium condition which provides the absence in the solution of conical points on the symmetry axis between the sources.

Introduction

In our recent short paper¹ we presented (in a surprisingly simple form) an exact 4-parametric family of static asymptotically flat solutions of electrovacuum Einstein - Maxwell equations which describes the equilibrium configurations of two (non-rotating) charged masses in General Relativity. In this paper we present a more general, 5-parametric solution of these equations which represent a nonlinear superposition of fields of two Reissner-Nordström sources with arbitrary mass and charge parameters and arbitrarily chosen separating distance and describe the procedure for a construction of a superposition of these fields. In the subsequent sections, we use a special divergent form of the reduced Einstein - Maxwell equations to derive the Komar-like integrals for the total gravitational mass and charge of this field and calculate the physical masses and charges of the sources defined as the additive inputs of each source in the total gravitational mass and total charge of the system. The expression of our solution in terms of these physical parameters simplifies it considerably. Then we determine the constraint which should be imposed on the parameters of this superposition of fields to provide the absence in the remaining 4-parametric solution of any non-physical singularities such as the conical points on the axis outside the sources. *This constraint plays the role of the condition for equilibrium of these sources in their common gravitational and electromagnetic fields.*

The 5-parametric solution, presented in this paper, was constructed using the monodromy transform approach^{2,3} which give rise to a reformulation of the Einstein - Maxwell equations for stationary axisymmetric fields in terms of equivalent system

of linear singular integral equations. As it is explained below, the functional parameters (monodromy data) in the kernels of these integral equations for our solution are chosen as analytically matched, rational functions of the spectral parameter. This choice of the monodromy data means that the solution possess *rational axis data*, i.e. that the values of all components of metric and electromagnetic potential, expressed in terms of cylindrical Weyl coordinates (ρ, z) , on the axis of symmetry $\rho = 0$ are rational functions of z . This means also that our solution virtually is the soliton solution on the Minkowski background in the sense, that it consists of a pure soliton part and its analytical continuation in the space of parameters.*

On the integral equation methods

Different approaches to solution of integrable space-time symmetry reductions of the Einstein's field equations, which began to develop about thirty years ago, gave rise to different formulations (generally non-equivalent to each other) of the integral equation methods, which allow to calculate the solutions of these field equations solving some systems of linear singular integral equations.

The first reformulation of symmetry reduced Einstein equations in terms of a system of linear singular integral equations for vacuum gravitational fields (together with a construction of vacuum solitons) was proposed in the framework of the *inverse scattering approach*⁴. This construction was based on a formulation of equivalent Riemann - Hilbert problem for the 2×2 -matrix functions on the spectral plane which gave rise to some system of linear singular integral equations. It is important, that the solutions of this equations possess the character of *solution generating transformations*, because the matrix kernel of these integral equations includes, as functional parameters, the components of an arbitrarily chosen vacuum metric which serves as the background for solitons, while the generating solution itself can be considered as describing some nonlinear perturbation of this background.

In the framework of another, *group-theoretic approach*, for construction of the solution generating transformations corresponding to the elements of the infinite dimensional algebra of internal symmetries of electrovacuum Einstein-Maxwell equations for stationary axisymmetric fields (found by Kinnersley and Chitre⁵), Hauser and Ernst⁶ reduced these equations to a homogeneous Hilbert problem for 3×3 -matrix functions of an auxiliary complex parameter and then, to the corresponding 3×3 -matrix linear singular integral equations. This construction assumed an additional constraint imposed on the class of solutions, which means that only those solutions are considered which possess (locally) a regular behaviour of fields near the axis of symmetry. The integral equations⁶ have rather complicate matrix kernel which construction includes a calculation of matrix exponents of the elements of Kinnersley and Chitre algebra, represented by some algebraically defined holomorphic 3×3 -matrix functions of the mentioned above auxiliary complex parameter.

*This analytical continuation is similar to that which connects different parts of the Kerr-Newman family of solutions corresponding to the field of a naked singularity and the black hole solution.

Later, fixing the choice of the seed solution by the simplest one, Sibgatullin⁷ reduced the matrix integral equations⁶ of Hauser and Ernst to a simpler scalar linear singular integral equation with some "normalization" condition imposed additionally on its solutions. The kernel of this scalar integral equation was expressed explicitly in terms of the values of the Ernst potentials on the axis of symmetry. This integral equation was actively used during the last one and a half decades by Sibgatullin and others for mostly formal calculation of asymptotically flat solutions for various particular choices of (asymptotically flat) rational axis data for the Ernst potentials.

The *monodromy transform approach*^{2,3} does not follow the ideology of the matrix Riemann-Hilbert problems, however, it is also based on some ideas of the modern theory of integrable systems, analytical theory of differential equations and the theory of linear singular integral equations. For physically different classes of vacuum and electrovacuum fields with two commuting isometries (stationary axisymmetric fields, plane and cylindrical waves, inhomogeneous cosmological solutions and some others) this approach suggests rather simple general construction of the coordinate-independent functional parameters (called as monodromy data) which characterize uniquely every local solution. The problem of constructing solutions for given monodromy data (the inverse problem of the monodromy transform) gave rise to a system of linear singular integral equations which differs essentially from the integral equations mentioned above. A specific structures of the (scalar) kernels of these integral equations and of the integration path on the spectral plane allow to describe all degrees of freedom of the gravitational and electromagnetic fields and make these integral equations *equivalent* to the symmetry reduced Einstein - Maxwell equations. For stationary axisymmetric fields satisfying the regularity axis condition, the integral equations^{2,3} simplify considerably. A general scheme for constructing of solutions with any *rational analytically matched monodromy data* (or, equivalently, of solutions for any rational, not only asymptotically flat, axis data) was described earlier in detail (see^{3,8} and the references therein).

Solitons on the Minkowski background and rational axis data.

A discovery of existence of pure gravitational solitons and of the ways for their generating on arbitrarily chosen vacuum backgrounds – the dressing methods⁴(see also⁹), together with initiated by these results developments of the similar methods for Einstein - Maxwell fields¹⁰, gave us a powerful tool for construction of a large variety of solutions of Einstein's field equations and for nonlinear superposition of fields of certain kinds of sources with various external fields.

On the other hand, later developments of the integral equation methods briefly described above suggested a simple idea to calculate the solutions of Einstein and Einstein - Maxwell equations using these integral equations with the simplest choice of functional parameters (the contour data, or the axis data for the Ernst potentials, or analytically matched monodromy data) in their kernels as rational functions of their arguments. This also leads to a construction of large families of exact solutions

with arbitrary (finitely large) number of free parameters, but without any freedom in the choice of the background solution. This gives rise to the obvious questions concerning a comparison of these two constructions.

First of all, it is clear, that the asymptotically flat rational-data solutions can be only a very special case of soliton solutions corresponding to a particular choice of the background for solitons. This can be the Minkowski space-time or any other soliton solution on this background. Indeed, a direct calculation shows, that vacuum solitons⁴ and electrovacuum solitons¹⁰, both generated on the Minkowski background, represent the asymptotically flat solutions with rational axis data. These solitons have as many of free parameters per one simple pole of the dressing matrix, as it is necessary to describe an arbitrary chosen rational asymptotically flat axis data with a twice lower number (for vacuum case) or with the same number of poles in this data.[†] Of course, the coincidence of the number of parameters itself does not mean that these classes of solutions coincide. Moreover, some doubts in this coincidence can arise from that fact that the construction¹⁰ of electrovacuum solitons (in contrast to vacuum soliton generating technique⁴) allows to generate the solitons with arbitrarily located complex poles of the dressing matrix and with the complex conjugated poles of the inverse matrix, while the solitons with real poles do not arise in this technique. Of course, one can try to obtain the electrovacuum solitons with real poles as limiting cases of solitons with complex poles, however, this leads to solutions with coinciding real poles of dressing matrix and its inverse, which have less number of free parameters per each pole than the solitons with complex poles.

To clarify this situation, it is worth to consider a one-soliton solution on the Minkowski background, which coincides with the over-extreme part of the well known Kerr-Newman solution corresponding to a naked singularity. The under-extreme part of this solution, which has a horizon and corresponds to a black hole, can be described in terms of the soliton generating technique by a dressing matrix with one real pole. However, as it is well known, the under-extreme part of the Kerr-Newman family can be obtained by a simple analytical continuation from its over-extreme part in the space of its parameters. It is very likely, that the similar analytical continuation is possible not only for a one-soliton solution on the Minkowski background, but for any number of solitons as well. Indeed, solving the integral equations for the (static) two-pole monodromy data, we have found that the part of our static 5-parametric solution, which corresponds to a pair of naked Reissner-Nordström singularities, coincides with the static subfamily of two-soliton solutions on the Minkowski background, while the remaining part of this family is connected with this two-soliton solution by the similar analytical continuations in the space of parameters as one uses in the one-soliton case to connect different parts of the Kerr-Newman family of solutions. The motivation given above shows

[†]It is necessary to note, however, that the poles of the dressing matrices for solitons coincide with the poles of the solutions of the integral equations, but they do not coincide in general, with the poles of the axis data for the Ernst potentials.

that there is the most reason to expect that the class of solutions of the integral equations for asymptotically flat rational axis data consists of the already known solitons on the Minkowski background together with their analytical continuations in the space of parameters (which should be supplied also, for completeness, by certain limiting solutions corresponding to multiple poles in the dressing matrices). On the one hand, in view of the above considerations, this means that there is no any reason to consider every formally calculated solution of the integral equations for rational asymptotically flat axis data as some new one describing the "extended" or "generalized" solitons, because it would be curiously enough to call similarly the under-extreme part of the Kerr-Newman family of solutions which is a simple analytically continuation in the parameter space of its over-extreme part.

On the other hand, the above motivation does not mean at all, that there is no sense to use the integral equation methods for calculation of these soliton solutions starting directly from some asymptotically flat rational axis data. It is worth to mention here, that the application of the soliton generating technique leads to some specific parameterization of the constructing solutions which differs significantly from that which arise for the solutions constructed from the corresponding solutions of the integral equations. Each of these parameterizations can occur to be more or less useful in different considerations of this class of soliton (or rational-data) solutions. For example, it seems, that just the existence of the analytical continuation in the parameter space discussed above can be proved for N-soliton solution more easily if we use its expression which arises from the solution of the integral equations, but considering the general twelve-parametric two-soliton solution, we find that it has the most compact form just in the soliton parameterization.³

This concludes our very brief reminding of some fragments of the history of the methods and of some interrelations between the solitons and solutions with rational axis data derived from the integral equation methods. At the end of this Introduction, we sketch out the monodromy transform approach and the integral equation method which we use for construction of our 5-parametric solution for a superposition of fields of two Reissner-Nordström sources. In all necessary details, a theory of this method and a general scheme for construction of solutions with rational, analytically matched monodromy data, have been developed long ago and described in the papers cited above with some additional useful references therein.

Monodromy transform approach

This approach is based a) on the parametrization of the space of local solutions of the symmetry reduced Einstein - Maxwell equations by the monodromy data – a set of coordinate independent functions of a spectral parameter, which determine the branching properties on the spectral plane of the fundamental solution of associated linear system, and b) on the reformulation of these equations in terms of a system of linear singular integral equations. For stationary axisymmetric fields, this approach allows to construct nonlinear superpositions of electrovacuum fields of

different sources characterized by analytically matched, rational monodromy data³ (see also the Appendix in⁸). In the following sections, we demonstrate at first that the external field of a single Reissner-Nordström source is characterized just by this kind of the monodromy data functions which have on the spectral plane one simple pole and vanish at infinity. For superposition of fields of two such sources we choose the rational monodromy data functions as the sums of two poles with such coefficients which guarantee that the solution is static. Given this monodromy data, we describe step by step the construction of the corresponding solution.

The space of local solutions and its parameterization by monodromy data

For electrovacuum Einstein - Maxwell fields depending only on two space-time coordinates, in the entire space of local solutions, which are analytical near some initial point and take at this point (the point of "normalization") some "standard" values, every local solution with the metric g_{ik} and electromagnetic potential A_i can be characterized uniquely by the monodromy data which consist of four coordinate-independent holomorphic functions of the spectral parameter w :

$$\{g_{ik}(x^1, x^2), A_i(x^1, x^2)\} \longleftrightarrow \{\mathbf{u}_{\pm}(w), \mathbf{v}_{\pm}(w)\} \quad (1)$$

This monodromy data are defined as a complete set of independent functions which characterize the branching properties of the corresponding fundamental solution of the associated linear system on the spectral plane at its four singular points. This data are defined uniquely for any local solution and for arbitrary choice of these functions there always exists a unique local solution of electrovacuum Einstein - Maxwell equations with given monodromy data.

This construction is useful as far as some effective methods can be developed for explicit constructing of local solutions for given monodromy data. Then, given an explicit local solution, the corresponding global solution can be determined using its analytical continuation to other space-time regions where this solution may reveal also some regular behaviour or approaches various types of singularities.

The master system of linear singular integral equations

The key point of the use of the monodromy data is the existence of some system of linear singular integral equations whose kernels and right hand sides are expressed algebraically in terms of the monodromy data and whose solution determines (by means of some quadratures) all components of metric and electromagnetic potential.

In general, the structure of this system of linear singular integral equations is rather complicate. The singular integrals are defined on the contour L on the spectral plane which consists of two disconnected parts $L = L_+ + L_-$. The locations of their endpoints depend on the space-time coordinates and coordinates of the initial point which enter also the integrands as parameters. In particular, L_+ goes from $w = \xi_o$ to $w = \xi$ and L_- goes from $w = \eta_o$ to $w = \eta$, where, for example, for stationary

axisymmetric fields in terms of Weyl coordinates $\xi = z + i\rho$, $\eta = z - i\rho$, but for plane waves $\xi = x + t$, $\eta = x - t$ and (ξ_o, η_o) correspond to the initial point.

The monodromy data as well as the unknown functions in these integral equations are defined in two disconnected regions of the spectral plane – the neighbourhoods of L_+ and L_- , where they are represented by pairs of functions $(\mathbf{u}_+, \mathbf{u}_-)$, $(\mathbf{v}_+, \mathbf{v}_-)$ and $(\boldsymbol{\varphi}_+, \boldsymbol{\varphi}_-)$ respectively. It is clear that these equations can not be solved for arbitrarily chosen monodromy data. However, for some classes of fields, such as, for example, stationary axisymmetric fields with a regular axis of symmetry considered in this paper, these integral equations can be simplified considerably and admit infinite hierarchies of multiparametric families of explicit solutions.

Monodromy data for stationary fields with a regular symmetry axis

For stationary axisymmetric fields, it is typical that physical and geometrical formulations of various problems (as, for example, in the case of asymptotically flat fields) imply that at least some part of the axis of symmetry is free of the field sources and therefore, a behaviour of metric and matter fields in the neighbourhood of this part of the axis should be regular. For such fields, if we choose the initial point (the point of normalization) of a solution on such regular part of the axis, the initial points of the contours L_+ and L_- coincide and instead of two disconnected contours we obtain one simple curve $L = L_+ + L_-$. Changing preliminary the direction of integration on L_- , we obtain one contour where the system of singular integral equations is defined. It starts at the point $w = \eta$, goes through the initial point $w = z_0$ and ends at $w = \xi$. On this contour the monodromy data and the unknown variable in the integral equations are represented by twice lower number of holomorphic functions because for these fields we have

$$\mathbf{u}_+(w) = \mathbf{u}_-(w) \equiv \mathbf{u}(w), \quad \mathbf{v}_+(w) = \mathbf{v}_-(w) \equiv \mathbf{v}(w), \quad \boldsymbol{\varphi}_+(w) = \boldsymbol{\varphi}_-(w) \equiv \boldsymbol{\varphi}(w) \quad (2)$$

Usually, we call these conditions as the regularity axis condition, however we note that these conditions guarantee only a regular local behaviour of fields near the axis, but they do not exclude the presence of some non-curvature singularities, such as conical points on the axis or closed time-like curves near it. Thus, for stationary axisymmetric electrovacuum fields with a regular axis of symmetry the monodromy data are represented only by two arbitrary holomorphic functions $\mathbf{u}(w)$ and $\mathbf{v}(w)$. We recall also that $\mathbf{v}(w)$ is "responsible" for a presence of electromagnetic field, so that for vacuum $\mathbf{v}(w) \equiv 0$ and the space of solutions of vacuum stationary axisymmetric fields near the regular part of the axis of symmetry is parameterized by the monodromy data consisting of one holomorphic function $\mathbf{u}(w)$.

Another important property of the stationary axisymmetric electrovacuum fields with the regular axis of symmetry is that any solution can be characterized by the finite values of its metric components and potentials (e.g., of the complex Ernst potentials $\mathcal{E}(\rho, z)$, $\Phi(\rho, z)$) on the regular part of the axis of symmetry. In this case,

the monodromy data (2) can be related to the values $\mathcal{E}(z)$, $\Phi(z)$ on the axis:

$$\mathcal{E}(\rho = 0, z) = \mathcal{E}_0 - 2i(z - z_0)\mathbf{u}(w = z), \quad \Phi(\rho = 0, z) = \Phi_0 + 2i(z - z_0)\mathbf{v}(w = z), \quad (3)$$

where z is the Weyl coordinate along the axis, z_0 is a coordinate of the initial point on this axis, \mathcal{E}_0 and Φ_0 are the "normalized" values of the Ernst potentials at the initial point, for which we usually put $\mathcal{E}_0 = 1$ and $\Phi_0 = 0$.

The conditions for asymptotically flat and static fields

The expressions (3) allow us to relate the structure of the monodromy data (2) with some physical and geometrical properties of fields. In particular, for any asymptotically flat field $\mathbf{u}(w)$ and $\mathbf{v}(w)$ should be holomorphic at $w = \infty$ and

$$\mathbf{u}(w) \rightarrow 0 \quad \text{and} \quad \mathbf{v}(w) \rightarrow 0 \quad \text{for} \quad w \rightarrow \infty.$$

Moreover, in this case, the coefficients of expansions of these functions in the inverse powers of w for $w \rightarrow \infty$ can be simply related to the multipole moments of this asymptotically flat field. Therefore, the multipole structure of the field can be determined in advance by the appropriate choice of the monodromy data.

For static fields, \mathcal{E} should be real, while Φ should be real for pure electric fields and imaginary for pure magnetic fields and therefore, $\mathbf{u}(w)$ and $\mathbf{v}(w)$ should satisfy

$$\mathbf{u}^\dagger(w) = -\mathbf{u}(w) \quad \text{and} \quad \mathbf{v}^\dagger(w) = \mp \mathbf{v}(w)$$

where $\mathbf{u}^\dagger(w) \equiv \overline{\mathbf{u}(\overline{w})}$ and $\mathbf{v}^\dagger(w) \equiv \overline{\mathbf{v}(\overline{w})}$ and a bar means a complex conjugation.

Exact solutions with rational monodromy data

An infinite hierarchies of solutions of the Einstein - Maxwell equations can be calculated explicitly if we choose the analytically matched monodromy data (2) to be rational functions of the spectral parameter:

$$\mathbf{u}(w) = \frac{U(w)}{Q(w)}, \quad \mathbf{v}(w) = \frac{V(w)}{Q(w)}$$

where the functions $U(w)$, $V(w)$ and $Q(w)$ are some polynomials. A general algorithm for solution of the integral equations for these polynomials of arbitrary orders was described in.^{3,8} This algorithm leads to explicit form of solutions in a unified, but rather complicate form, and, as we shall see below, a large careful work is necessary for finding of appropriate choice of physical parameters which can simplify significantly the constructed solutions.

The linear singular integral equations

For stationary axisymmetric fields outside their sources the metric and electromagnetic vector potential can be considered in cylindrical coordinates in the form

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 - f(d\rho^2 + dz^2) \\ A_i &= \{A_t, 0, 0, A_\varphi\}, \end{aligned} \quad (4)$$

where $x^i = \{t, \rho, z, \varphi\}$ and the metric components g_{tt} , $g_{t\varphi}$, $g_{\varphi\varphi}$ and f as well as the components A_t and A_φ of the electromagnetic potential are functions of the coordinates ρ and z only. It is well known, that for the metric and electromagnetic fields (4) the symmetry reduced electrovacuum Einstein - Maxwell field equations decouple into two parts. One of these parts is a closed system of "dynamical" equations - the nonlinear partial differential equations for the functions g_{tt} , $g_{t\varphi}$, $g_{\varphi\varphi}$, A_t and A_φ . (We recall here that the Weyl cylindrical coordinates are defined so that $g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = -\rho^2$ and therefore, only two of these three metric components are unknown functions). Another part is a pair of constraint equations which allow to determine in quadratures the conformal factor f provided the solution for g_{tt} , $g_{t\varphi}$, $g_{\varphi\varphi}$, A_t and A_φ is already known. Though an explicit calculation of these quadratures for the conformal factor f represent usually a large technical difficulty, the principal problem is a construction of the solution of the "dynamical" equations with wanted physical and geometrical properties. That is why we concentrate below mainly on the construction of solution of the "dynamical" equations.

In accordance with a general scheme,^{2,3} the components of metric and electromagnetic potential (4), which satisfy the electrovacuum Einstein - Maxwell equations, can be expressed in the form

$$\begin{aligned} g_{tt} &= 1 - i(R_t^\varphi - \bar{R}_t^\varphi) + \Phi_t \bar{\Phi}_t & \mathcal{E} &= 1 - 2iR_t^\varphi \\ g_{t\varphi} &= -i(z - z_0) + i(R_t^t + \bar{R}_\varphi^\varphi) + \Phi_t \bar{\Phi}_\varphi & \begin{pmatrix} \Phi_t \\ \Phi_\varphi \end{pmatrix} &= 2i \begin{pmatrix} R_t^* \\ R_\varphi^* \end{pmatrix} \\ g_{\varphi\varphi} &= i(R_\varphi^t - \bar{R}_\varphi^t) + \Phi_\varphi \bar{\Phi}_\varphi \end{aligned} \quad (5)$$

where $\{\Phi_t, \Phi_\varphi\}$ are the components of a complex electromagnetic potential and $\text{Re}\Phi_t = A_t$, $\text{Re}\Phi_\varphi = A_\varphi$; the functions $R_t^t, R_t^\varphi, R_t^*, R_\varphi^t, R_\varphi^\varphi, R_\varphi^*$ constitute a matrix which is determined by the integral over the contour L on the spectral plane

$$\mathbf{R} \equiv \begin{pmatrix} R_t^t & R_t^\varphi & R_t^* \\ R_\varphi^t & R_\varphi^\varphi & R_\varphi^* \end{pmatrix} = \frac{1}{i\pi} \int_L [\lambda]_\zeta \begin{pmatrix} 1 + i(\zeta - z_0)\mathbf{u}^\dagger(\zeta) \\ -i(\zeta - z_0) \end{pmatrix} \otimes \{\boldsymbol{\varphi}^{[1]}(\zeta), \boldsymbol{\varphi}^{[u]}(\zeta), \boldsymbol{\varphi}^{[v]}(\zeta)\} d\zeta \quad (6)$$

Here $\zeta \in L$ and the contour L (unlike the general case) is a simple curve which starts from $w = \eta \equiv z - i\rho$, goes through the initial point $w = z_0$ and ends at the point $w = \xi \equiv z + i\rho$, where ρ and z are the well known cylindrical Weyl coordinates in which $g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = -\rho^2$, and z_0 determines the location of the point of normalization on the axis of symmetry $\rho = 0$. In the above integral, $[\lambda]_\zeta$ denotes a jump (i.e. a half of the difference between left and right limits) at the point $\zeta \in L$ of a "standard" branching function

$$\lambda = \sqrt{(\zeta - z - i\rho)(\zeta - z + i\rho)/(\zeta - z_0)^2}, \quad \lambda(\rho, z, \zeta = \infty) = 1. \quad (7)$$

The functions $\boldsymbol{\varphi}^{[1]}(\zeta)$, $\boldsymbol{\varphi}^{[u]}(\zeta)$, $\boldsymbol{\varphi}^{[v]}(\zeta)$ should satisfy to the decoupled linear singular integral equations with the same scalar kernels and different right hand sides:^{2,3}

$$-\frac{1}{\pi i} \oint_L \frac{[\lambda]_\zeta \mathcal{H}(\tau, \zeta)}{\zeta - \tau} \begin{pmatrix} \boldsymbol{\varphi}^{[1]}(\zeta) \\ \boldsymbol{\varphi}^{[u]}(\zeta) \\ \boldsymbol{\varphi}^{[v]}(\zeta) \end{pmatrix} d\zeta = \begin{pmatrix} 1 \\ \mathbf{u}(\tau) \\ \mathbf{v}(\tau) \end{pmatrix} \quad (8)$$

where $\zeta, \tau \in L$; there is a Cauchy principal value integral in the left hand side, and the kernel function $\mathcal{H}(\tau, \zeta)$ in its integrand is

$$\mathcal{H}(\tau, \zeta) = 1 + i(\zeta - z_0)[\mathbf{u}^\dagger(\zeta) - \mathbf{u}(\tau)] + 4(\zeta - z_0)^2 \mathbf{v}(\tau) \mathbf{v}^\dagger(\zeta)$$

and everywhere below, we can put $z_0 = 0$ without any loss of generality. The coordinates ρ and z enter the equations (8) as the parameters which determine the location of the endpoints of the contour L and as the arguments of the function λ , however, for simplicity we have not shown explicitly in (8) the dependence of the unknown functions $\boldsymbol{\varphi}^{[1]}$, $\boldsymbol{\varphi}^{[u]}$ and $\boldsymbol{\varphi}^{[v]}$ on these coordinates. We note also, that in general, $\boldsymbol{\varphi}^{[1]}$, $\boldsymbol{\varphi}^{[u]}$ and $\boldsymbol{\varphi}^{[v]}$ as well as the right hand sides of (8) constitute the row-vectors, however, for a convenience we write (8) in a transposed form.

The Kerr - Newman field as a one-pole solution

We begin our description of solution of the integral equation (8) with a simple case of a one-pole structure of the monodromy data functions:

$$\mathbf{u}(w) = \frac{u_0}{w - h}, \quad \mathbf{v}(w) = \frac{v_0}{w - h}. \quad (9)$$

where u_0 , v_0 and h are arbitrary complex constants. For these data the kernel function $\mathcal{H}(\tau, \zeta)/(\zeta - \tau)$ can be split into the singular and regular parts:

$$\frac{\mathcal{H}(\tau, \zeta)}{\zeta - \tau} = \frac{1}{(\tau - h)(\zeta - \bar{h})} \left[\frac{P(\zeta)}{\zeta - \tau} + R(\zeta) \right] \quad (10)$$

where the polynomials $P(\zeta)$ and $R(\zeta)$ possess the expressions ($z_0 = 0$):

$$\begin{aligned} P(\zeta) &= \zeta^2 [1 - i(u_0 - \bar{u}_0) + 4v_0 \bar{v}_0] - \zeta [(1 + i\bar{u}_0)h + (1 - iu_0)\bar{h}] + h\bar{h} \\ R(\zeta) &= -(1 + i\bar{u}_0)\zeta + \bar{h} \end{aligned} \quad (11)$$

Assuming that $h \neq 0$, what means that the pole is not located on the integration path L , we present the polynomial $P(w)$ in a factorized form

$$P(w) = P_0(w - w_1)(w - \tilde{w}_1), \quad P_0 = 1 - i(u_0 - \bar{u}_0) + 4v_0 \bar{v}_0 \quad (12)$$

The coefficients of $P(w)$ always are real and therefore, its roots w_1 and \tilde{w}_1 are real or complex conjugated to each other. We parameterize these roots as

$$w_1 = z_1 + \sigma_1, \quad \tilde{w}_1 = z_1 - \sigma_1 \quad (13)$$

where z_1 is a real parameter while σ_1 can be real or pure imaginary. A comparison of (11) and (12) allows to express z_1 and σ_1 in terms of u_0 , v_0 and h , however, it is more convenient to use z_1 and σ_1 as new parameters and express some of the parameters u_0 , v_0 and h as functions of z_1 , σ_1 and others. We give the explicit expressions later, but now we concentrate on solving of the integral equations (8).

As it can be concluded from the structure of the singular integral equations, their solutions for the monodromy data (9) should have the form

$$\left\{ \boldsymbol{\varphi}^{[1]}(w), \boldsymbol{\varphi}^{[u]}(w), \boldsymbol{\varphi}^{[v]}(w) \right\} = \frac{(w - \bar{h})}{P(w)} \left\{ w + X_0, Y_0, Z_0 \right\} \quad (14)$$

where X_0 , Y_0 and Z_0 are independent of the spectral parameter w , but they can depend on the coordinates ρ and z . As we see from these expressions, the solutions of the integral equations (8) are rational functions of the spectral parameter and they have the poles coinciding with the roots of the polynomial $P(w)$.

To calculate the values of X_0 , Y_0 and Z_0 explicitly, we substitute (9), (10) and (14) into the integral equations (8) and obtain the linear algebraic equations for X_0 , Y_0 and Z_0 with rather complicate coefficients. These coefficients are linear combinations of the usual or singular Cauchy principal value integrals of the form

$$\frac{1}{\pi i} \oint_L \frac{[\lambda]_\zeta \zeta^k}{\zeta - \tau} d\zeta, \quad \frac{1}{\pi i} \int_L \frac{[\lambda]_\zeta \zeta^k}{P(\zeta)} d\zeta$$

where $\tau, \zeta \in L$ and $k \geq 0$ is some integer. Any integral of these types can be calculated explicitly using the elementary theory of residues. Indeed, the integrands in these integrals can be expressed as the jumps of analytical functions $\lambda(w) w^k / (w - \tau)$ and $\lambda(w) w^k / P(w)$ respectively. Therefore, these integrals can be expressed as the integrals over the closed curves \mathcal{L} surrounding the contour L :

$$\frac{1}{\pi i} \oint_L \frac{[\lambda]_\zeta \zeta^k}{\zeta - \tau} d\zeta = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\lambda(\chi) \chi^k}{\chi - \tau} d\chi, \quad \frac{1}{\pi i} \int_L \frac{[\lambda]_\zeta \zeta^k}{P(\zeta)} d\zeta = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\lambda(\chi) \chi^k}{P(\chi)} d\chi \quad (15)$$

where $\chi \in \mathcal{L}$ and \mathcal{L} surrounds L in negative direction (i.e. so that the interior is to the right) and close enough to L so that no poles of $P(w)$ are inside \mathcal{L} . The function $\lambda(w) w^k / (w - \tau)$ is analytical outside the contour L and it can have only the pole at $w = \infty$, while the function $\lambda(w) w^k / P(w)$ is analytical outside L , besides the poles which arise from zeros of $P(w)$ and possible poles at $w = \infty$. This allows us to transform the closed integration path \mathcal{L} into a one approaching $w = \infty$ taking into account the inputs from the finite poles and therefore, in (15), each of the integrals over \mathcal{L} is equal to the sum of residues of its integrand at finite poles plus the residue at $w = \infty$. Thus, for the integrals (15) we obtain

$$\begin{aligned} \frac{1}{\pi i} \oint_L \frac{[\lambda]_\zeta \zeta^k}{\zeta - \tau} d\zeta &= - \sum_{m=0}^k (\lambda)_{k-m} \tau^m, \\ \frac{1}{\pi i} \int_L \frac{[\lambda]_\zeta \zeta^k}{P(\zeta)} d\zeta &= \frac{\lambda(w_1) w_1^k}{P'(w_1)} + \frac{\lambda(\tilde{w}_1) \tilde{w}_1^k}{P'(\tilde{w}_1)} - \left(\frac{\lambda(w) w^k}{P(w)} \right)_{-1} \end{aligned} \quad (16)$$

where $(\lambda)_{m-k}$ means the coefficient in front of $1/w^{k-m}$ in the inverse powers expansion of the function $\lambda(w)$ at $w = \infty$ and similarly, $(\dots)_{-1}$ means the coefficient in front of $1/w$ in the inverse powers expansion of a function at $w = \infty$. The parameters w_1 and \tilde{w}_1 are the roots of the polynomial $P(w)$, while $P'(w_1)$ and $P'(\tilde{w}_1)$ mean a derivative of $P(w)$ with respect to w at $w = w_1$ or $w = \tilde{w}_1$ respectively.

The substitution of (14) and (10) into (8) and the subsequent calculation of the contour integrals leads to the equations whose left and right hand sides, after multiplication by $(\tau - h)$, become polynomial functions of τ . Equating the coefficients

of these polynomials, we obtain the linear algebraic equations for X_0 , Y_0 and Z_0 whose solution allows to obtain the explicit expressions for $\boldsymbol{\varphi}$:

$$\left\{ \boldsymbol{\varphi}^{[1]}(w), \boldsymbol{\varphi}^{[u]}(w), \boldsymbol{\varphi}^{[v]}(w) \right\} = \frac{(w - \bar{h})}{P(w)} \left\{ w + z - \frac{h + \Delta_1}{\Delta_0}, \frac{u_0}{\Delta_0}, \frac{v_0}{\Delta_0} \right\} \quad (17)$$

where Δ_0 and Δ_1 possess the expressions

$$\begin{aligned} \Delta_0 &= -\frac{\lambda(w_1)}{P'(w_1)} R(w_1) - \frac{\lambda(\tilde{w}_1)}{P'(\tilde{w}_1)} R(\tilde{w}_1) + 1 - \frac{1 + i\bar{u}_0}{P_0} \\ \Delta_1 &= -\frac{\lambda(w_1)}{P'(w_1)} R(w_1)(w_1 + z) - \frac{\lambda(\tilde{w}_1)}{P'(\tilde{w}_1)} R(\tilde{w}_1)(\tilde{w}_1 + z) \\ &\quad - \frac{1 + i\bar{u}_0}{P_0^2} [\bar{h}(1 - iu_0) + h(1 + i\bar{u}_0)] + \frac{\bar{h}}{P_0}. \end{aligned} \quad (18)$$

and the polynomials $P(\zeta)$ and $R(\zeta)$ were defined in (11). Now we have to substitute these solutions of the integral equations into the integrands of quadratures (6) and calculate these quadratures using the same methods for calculation of these type of integrals as it was described above. The components of metric and of electromagnetic potential, as well as the Ernst potentials for the constructing solution can be calculated then algebraically from the expressions (5).

The formal calculations for the monodromy data (9) described above lead us to the explicit but very complicate form of the solution. This solution can be simplified considerably using (i) a new and most appropriate set of parameters, (ii) some global gauge transformations – $SL(2, R)$ -rotations and rescalings of the Killing vectors ξ_t and ξ_φ with the corresponding linear transformations of t and φ coordinates and (iii) a set of more convenient coordinates (such as the prolate or oblate ellipsoidal coordinates) instead of the Weyl coordinates ρ and z . In the remaining part of this section we describe these simplifications and show that this solution represents nothing more but the well known Kerr - Newman family.

At first, comparing the explicit expression (11) for $P(w)$ with its factorized form (12) and using (13), we obtain a set of relations between z_1 , σ_1 and the original set of independent parameters u_0 , v_0 and h . Solving these relations we express u_0 , v_0 and h in terms of a new set of parameters which consists of the real parameters m , a , b , z_1 and a complex parameter e :

$$\begin{aligned} u_0 &= -i + \frac{ih}{w_1 \tilde{w}_1} (z_1 + ia), & h &= z_1 + m - i(a + b) \\ v_0 &= \frac{ie}{2h} \sqrt{P_0}, & P_0 &= \frac{h\bar{h}}{w_1 \tilde{w}_1}, \end{aligned} \quad (19)$$

and the parameter σ_1 is related to the parameters m , a , b , z_1 and e by the equation

$$\sigma_1^2 = m^2 + b^2 - a^2 - e\bar{e}$$

Now, for the choice of new coordinates we note, that besides a presence of ρ and z in the solution explicitly, its dependence of these coordinates also comes from the

functions $\lambda(\rho, z, w)$ which should be calculated at the points $w = w_1$ and $w = \tilde{w}_1$, i.e. at the roots of $P(w)$. Let us express these functions in the form

$$\lambda(\rho, z, w_1) = \frac{x_1 - \sigma_1 y_1}{z_1 + \sigma_1}, \quad \lambda(\rho, z, \tilde{w}_1) = \frac{x_1 + \sigma_1 y_1}{z_1 - \sigma_1}$$

where x_1 and y_1 will be considered as new coordinates. If we compare these expressions with the definition (7), we obtain the relations between ρ, z and x_1, y_1 :

$$\rho = \sqrt{x_1^2 - \sigma_1^2} \sqrt{1 - y_1^2}, \quad z = z_1 + x_1 y_1$$

Finally, we have to make some gauge transformation of our solution:

$$\begin{aligned} h_{tt} &\rightarrow \frac{h_{tt}}{P_0} & \mathcal{E} &\rightarrow 1 + \frac{\mathcal{E} - \mathcal{E}_0 + 2\bar{\Phi}_0(\Phi - \Phi_0)}{P_0} & \mathcal{E}_0 &= 1 - 2iu_0 \\ h_{t\varphi} &\rightarrow h_{t\varphi} & \Phi &\rightarrow \frac{\Phi - \Phi_0}{\sqrt{P_0}} & \Phi_0 &= 2iv_0 \\ h_{\varphi\varphi} &\rightarrow P_0 h_{\varphi\varphi} & \tilde{\Phi} &\rightarrow \sqrt{P_0} \tilde{\Phi} \end{aligned}$$

After this reparametrization, change of coordinates and gauge transformations we obtain the solution in a simple form. In particular, we obtain the Ernst potentials

$$\begin{aligned} \mathcal{E} &= 1 - 2 \frac{(-i\epsilon_0 u_0 + 4\epsilon_0 v_0 v_0^\dagger)}{P_0 \Delta_0} = 1 - \frac{2(m - ib)}{x_1 + iay_1 + m - ib}, \\ \Phi &= -\frac{2iv_0}{\sqrt{P_0} \Delta_0} = \frac{e}{x_1 + iay_1 + m - ib} \end{aligned}$$

which can be identified with those for the well known Kerr-Newman solution. The coordinates x_1 and y_1 are connected directly with the polar spheroidal coordinates:

$$x_1 = r_1 - m, \quad y_1 = \cos \theta_1$$

and the real parameters m, a, b, z_1 , real and imaginary part of e can be identified respectively with the mass, angular momentum, NUT-parameter, electric and magnetic charges of the Kerr-Newman source (a black hole or a naked singularity).

The Reissner - Nordström field as a static one-pole solution

As it was already mentioned above, to obtain a static solution with the monodromy functions having only one pole, it is not necessary to make all of the calculations described above, but it would be enough to restrict from the beginning the monodromy data functions by the case, when these functions would take pure imaginary values on the real axis on the spectral plane. For this, it is necessary to choose the parameter h to be real and the parameters u_0 and v_0 pure imaginary. In this case, as one can see from (19), we would have

$$h = \bar{h}, \quad u_0 = -\overline{u_0}, \quad v_0 = -\overline{v_0} \quad \Longleftrightarrow \quad a = 0, \quad b = 0, \quad e = \bar{e}$$

i.e. we obtain the Reissner - Nordström solution.

Superposing the fields of two Reissner - Nordström sources

In this section we describe the key points of the calculation of solutions for monodromy data having two simple poles on the spectral plane. This monodromy data can be represented as a superposition of two one-pole terms and therefore, in accordance with the previous section, it is naturally to expect that the corresponding solution will describe the superposition of fields of two Reissner - Nordström sources.

Structure of the monodromy data

Thus, we begin with the choice of the monodromy data functions in the form

$$\mathbf{u}(w) = \frac{u_1}{w - h_1} + \frac{u_2}{w - h_2}, \quad \mathbf{v}(w) = \frac{v_1}{w - h_1} + \frac{v_2}{w - h_2}. \quad (20)$$

where u_1, u_2, v_1, v_2, h_1 and h_2 are arbitrary complex constants. However, to obtain a static solution, we specify this data imposing the following constraints:

$$\begin{aligned} u_1 &= -\overline{u_1} & v_1 &= -\overline{v_1} & h_1 &= \overline{h_1} \\ u_2 &= -\overline{u_2} & v_2 &= -\overline{v_2} & h_2 &= \overline{h_2} \end{aligned} \quad (21)$$

Structure of the kernel and a new set of parameters

In this case, the kernel $\mathcal{H}(\tau, \zeta)/(\zeta - \tau)$ splits into the singular and regular parts:

$$\frac{\mathcal{H}(\tau, \zeta)}{\zeta - \tau} = \frac{1}{(\tau - h)(\zeta - \overline{h})} \left[\frac{P(\zeta)}{\zeta - \tau} + R(\tau, \zeta) \right]$$

where the polynomials $P(\zeta)$, $R(\tau, \zeta)$ are defined as (everywhere below $z_0 = 0$):

$$\begin{aligned} P(\zeta) &= h_1^2 h_2^2 - 2h_1 h_2 [(1 - iu_1)h_2 + (1 - iu_2)h_1] \zeta + [(h_1 + h_2)^2 + 2h_1 h_2 \\ &\quad - 4(h_1 v_2 + h_2 v_1)^2 - 2i(h_1^2 u_2 + h_2^2 u_1) - 4ih_1 h_2 (u_1 + u_2)] \zeta^2 \\ &\quad + [2h_1(-1 + iu_1 + 2iu_2 + 4v_1 v_2 + 4v_2^2) + 2h_2(-1 + 2iu_1 + iu_2 \\ &\quad + 4v_1 v_2 + 4v_1^2)] \zeta^3 + [1 - 2i(u_1 + u_2) - 4(v_1 + v_2)^2] \zeta^4 \\ R(\tau, \zeta) &= R_0(\zeta) + R_1(\zeta) \tau \\ R_0(\zeta) &= h_1 h_2 (h_1 + h_2) \\ &\quad - [h_1^2 + 3h_1 h_2 + h_2^2 - ih_1(h_1 + 2h_2)u_2 - ih_2(2h_1 + h_2)u_1] \zeta \\ &\quad + [2(h_1 + h_2)(1 - iu_1 - iu_2 - 2v_1 v_2) - ih_1 u_2 - ih_2 u_1 - 4h_1 v_2^2 - 4h_2 v_1^2] \zeta^2 \\ &\quad + [-1 + 2i(u_1 + u_2) + 4(v_1 + v_2)^2] \zeta^3 \\ R_1(\zeta) &= -h_1 h_2 + (h_1 + h_2 - ih_1 u_2 - ih_2 u_1) \zeta + (-1 + iu_1 + iu_2) \zeta^2 \end{aligned} \quad (22)$$

Assuming that the pole of the monodromy data is not located on the integration path L and therefore, $h_1 h_2 \neq 0$, we present $P(w)$ in a factorized form

$$P(w) = P_0(w - w_1)(w - \tilde{w}_1)(w - w_2)(w - \tilde{w}_2) \quad (23)$$

The coefficients of the polynomial $P(w)$ always are real and therefore, its roots w_1 , \tilde{w}_1 and w_2 , \tilde{w}_2 are real or complex conjugated in pairs. We parameterize them as

$$\begin{aligned} w_1 &= z_1 + \sigma_1 & w_2 &= z_2 + \sigma_2 & h_1 &= z_1 + \tilde{m}_1 & z_2 &= z_1 + \ell \\ \tilde{w}_1 &= z_1 - \sigma_1 & \tilde{w}_2 &= z_2 - \sigma_2 & h_2 &= z_2 + \tilde{m}_2 \end{aligned} \quad (24)$$

where z_1 and z_2 (and therefore, ℓ) are real parameters while σ_1 as well as σ_2 can be real or pure imaginary. A comparison of coefficients of different powers of the spectral parameter in (22) and in (23) leads to a number of relations of the form

$$\begin{aligned} h_1^2 h_2^2 &= P_0 \mathcal{I}_4, \\ 2h_1 h_2 (h_1 + h_2 - ih_1 u_2 - ih_2 u_1) &= P_0 \mathcal{I}_3, \\ h_1^2 + h_2^2 + 4h_1 h_2 - 2i(h_1 + h_2)(h_1 u_2 + h_2 u_1) - 2ih_1 h_2 (u_1 + u_2) \\ &\quad - 4(h_1 v_2 + h_2 v_1)^2 = P_0 \mathcal{I}_2, \\ 2(h_1 + h_2) - 2i(h_1 u_2 + h_2 u_1) - 2i(h_1 + h_2)(u_1 + u_2) \\ &\quad - 8(h_1 v_2 + h_2 v_1)(v_1 + v_2) = P_0 \mathcal{I}_1, \\ 1 - 2i(u_1 + u_2) - 4(v_1 + v_2)^2 &= P_0. \end{aligned} \quad (25)$$

where we have introduced the notations

$$\begin{aligned} \mathcal{I}_1 &= w_1 + \tilde{w}_1 + w_2 + \tilde{w}_2 \\ \mathcal{I}_2 &= w_1 \tilde{w}_1 + w_1 w_2 + w_1 \tilde{w}_2 + \tilde{w}_1 w_2 + \tilde{w}_1 \tilde{w}_2 + w_2 \tilde{w}_2 \\ \mathcal{I}_3 &= \tilde{w}_1 w_2 \tilde{w}_2 + w_1 w_2 \tilde{w}_2 + w_1 \tilde{w}_1 \tilde{w}_2 + w_1 \tilde{w}_1 w_2 \\ \mathcal{I}_4 &= w_1 \tilde{w}_1 w_2 \tilde{w}_2 \end{aligned}$$

In terms of the parameters (24) these functions possess the expressions:

$$\begin{aligned} \mathcal{I}_1 &= 2(z_1 + z_2) & \mathcal{I}_3 &= 2z_1 z_2 (z_1 + z_2) - 2z_2 \sigma_1^2 - 2z_1 \sigma_2^2 \\ \mathcal{I}_2 &= z_1^2 + 4z_1 z_2 + z_2^2 - \sigma_1^2 - \sigma_2^2 & \mathcal{I}_4 &= (z_1^2 - \sigma_1^2)(z_2^2 - \sigma_2^2) \end{aligned}$$

The relations (25) allow to express the roots of $P(w)$ (or the new parameters z_1 , z_2 and σ_1 , σ_2 in terms of u_1 , u_2 , v_1 , v_2 and h_1 , h_2 , however, it is more simple and convenient to use z_1 , z_2 and σ_1 , σ_2 as new parameters and express P_0 and some of the parameters u_1 , u_2 , v_1 , v_2 and h_1 , h_2 as functions of z_1 , z_2 , σ_1 , σ_2 and others. We present these expressions in the form

$$\begin{aligned} u_1 &= \frac{2ih_1 h_2^2 \tilde{e}_1 \tilde{e}_2 P_1 P_2 + 2ih_1^3 \tilde{e}_2^2 P_2^2 + ih_1^2 (h_1 - h_2)(h_2^3 \mathcal{I}_1 - 2h_2^2 \mathcal{I}_2 + 3h_2 \mathcal{I}_3 - 4\mathcal{I}_4)}{2(h_1 - h_2)^3 \mathcal{I}_4}, \\ u_2 &= \frac{-2ih_1^2 h_2 \tilde{e}_1 \tilde{e}_2 P_1 P_2 - 2ih_2^3 \tilde{e}_1^2 P_1^2 + ih_2^2 (h_1 - h_2)(h_1^3 \mathcal{I}_1 - 2h_1^2 \mathcal{I}_2 + 3h_1 \mathcal{I}_3 - 4\mathcal{I}_4)}{2(h_1 - h_2)^3 \mathcal{I}_4}, \\ v_1 &= -\frac{ih_2 \tilde{e}_1 P_1}{2(h_1 - h_2)\sqrt{\mathcal{I}_4}}, & v_2 &= -\frac{ih_1 \tilde{e}_2 P_2}{2(h_1 - h_2)\sqrt{\mathcal{I}_4}}. \end{aligned} \quad (26)$$

where the parameters \tilde{e}_1 , \tilde{e}_2 and P_1 , P_2 are defined as follows. Solving (25) leads to the expressions $P(h_1)$ and $P(h_2)$ which must be non-negative:

$$\begin{aligned} P(h_1) &\equiv (h_1 - w_1)(h_1 - \tilde{w}_1)(h_1 - w_2)(h_1 - \tilde{w}_2) \geq 0 \\ P(h_2) &\equiv (h_2 - w_1)(h_2 - \tilde{w}_1)(h_2 - w_2)(h_2 - \tilde{w}_2) \geq 0 \end{aligned} \quad (27)$$

Assuming for the case of real roots of $P(w)$ that they are numbered so that $\tilde{w}_1 < w_1 < \tilde{w}_2 < w_2$, and that in any case, $z_1 < z_2$, we conclude from (27) that h_1 and

h_2 should be located outside the interval (\tilde{w}_1, w_1) if w_1 (and therefore, \tilde{w}_1) is real and outside the interval (\tilde{w}_2, w_2) if w_2 (and therefore, \tilde{w}_2) is real. This means that

$$(h_1 - w_1)(h_1 - \tilde{w}_1) \equiv \tilde{m}_1^2 - \sigma_1^2 \geq 0, \quad (h_2 - w_2)(h_2 - \tilde{w}_2) \equiv \tilde{m}_2^2 - \sigma_2^2 \geq 0$$

and therefore, there exist the real parameters \tilde{e}_1 and \tilde{e}_2 such that

$$\tilde{e}_1^2 = \tilde{m}_1^2 - \sigma_1^2, \quad \tilde{e}_2^2 = \tilde{m}_2^2 - \sigma_2^2 \quad (28)$$

and we allow to each of the parameters \tilde{e}_1 and \tilde{e}_2 to be positive or negative in order to cover all opportunities to have in the expressions different signs in front of the square roots: $\pm\sqrt{P(h_1)}$ and $\pm\sqrt{P(h_2)}$. The parameters P_1 and P_2 in (26) are defined as positive parameters which have the following explicit expressions:

$$\begin{aligned} P_1 &= \sqrt{(h_1 - w_2)(h_1 - \tilde{w}_2)} = \sqrt{(\ell - \tilde{m}_1)^2 - \sigma_2^2} \\ P_2 &= \sqrt{(h_2 - w_1)(h_2 - \tilde{w}_1)} = \sqrt{(\ell + \tilde{m}_2)^2 - \sigma_1^2} \end{aligned} \quad (29)$$

The expressions (26) allow us to use instead u_0, u_1, v_0, v_1 and h_1, h_2 the parameters $z_1, z_2, \tilde{m}_1, \tilde{m}_2, \tilde{e}_1, \tilde{e}_2$ (with $\ell = z_2 - z_1 > 0$) which can take arbitrary real values, provided $P_1^2 > 0$ and $P_2^2 > 0$, in view of (29) and (28). Without loss of generality, we assume that if all roots are real, then $\sigma_1 > 0, \sigma_2 > 0$, and for the sources to be separated, we assume also that $\ell > \sigma_1 + \sigma_2$.

Formal construction of the solution

For solving of the integral equations (8) for the monodromy data (20) - (21), we substitute this data into the integral equation (8) and obtain the equations whose solution has the structure which is very similar to that for the one-pole case (14):

$$\begin{pmatrix} \boldsymbol{\varphi}^{[1]}(w) \\ \boldsymbol{\varphi}^{[u]}(w) \\ \boldsymbol{\varphi}^{[v]}(w) \end{pmatrix} = \frac{(w - h_1)(w - h_2)}{P(w)} \begin{pmatrix} X_0 + X_1 w + X_2 w^2 \\ Y_0 + Y_1 w + Y_2 w^2 \\ Z_0 + Z_1 w + Z_2 w^2 \end{pmatrix} \quad (30)$$

where the coefficients X_k, Y_k and Z_k ($k = 0, 1, 2$) are independent of the spectral parameter w and they are functions of coordinates and constant parameters. To find these coefficients explicitly, we have to substitute (30) back into the integral equations, to calculate the corresponding integrals using the rules (16) and solve the linear algebraic equations for X_k, Y_k and Z_k . Using this solution, we find explicitly (using again the rules (16)) all components (6) of the matrix \mathbf{R} and calculate then pure algebraically the Ernst potentials and all metric and electromagnetic potential components in accordance with their general expressions (5).

Weyl cylindrical and bipolar coordinates

During the calculation of the contour integrals (15), the coordinate dependence of the solution arises from calculation of residues of the integrands. It is easy to see, that the residue at infinity gives rise to the terms which are polynomial functions of Weyl coordinates ρ and z , while the residues at finite poles — the roots of $P(w)$ give

rise to the terms which are proportional to the values of the function $\lambda(\xi, \eta, w)$ at these roots: $\lambda_1 = \lambda(\xi, \eta, w_1)$, $\tilde{\lambda}_1 = \lambda(\xi, \eta, \tilde{w}_1)$, $\lambda_2 = \lambda(\xi, \eta, w_2)$ and $\tilde{\lambda}_2 = \lambda(\xi, \eta, \tilde{w}_2)$. To have deal with these functions, we introduce bipolar coordinates (x_1, y_1) and (x_2, y_2) related to the corresponding pairs of spherical-like coordinates (r_1, θ_1) and (r_2, θ_2) centered on the symmetry axis $\rho = 0$ at $z = z_1$ and $z = z_2$, so that $x_1 = r_1 - \tilde{m}_1$, $y_1 = \cos \theta_1$, $x_2 = r_2 - \tilde{m}_2$, $y_2 = \cos \theta_2$. The coordinates (x_1, y_1) and (x_2, y_2) are defined by the relations to λ_1 , $\tilde{\lambda}_1$, λ_2 and $\tilde{\lambda}_2$

$$\begin{aligned} \lambda_1 &= \frac{x_1 - \sigma_1 y_1}{z_1 + \sigma_1} & \tilde{\lambda}_1 &= \frac{x_1 + \sigma_1 y_1}{z_1 - \sigma_1} & \begin{cases} \rho = \sqrt{x_1^2 - \sigma_1^2} \sqrt{1 - y_1^2} \\ z = z_1 + x_1 y_1 \end{cases} \\ \lambda_2 &= \frac{x_2 - \sigma_2 y_2}{z_2 + \sigma_2} & \tilde{\lambda}_2 &= \frac{x_2 + \sigma_2 y_2}{z_2 - \sigma_2} & \begin{cases} \rho = \sqrt{x_2^2 - \sigma_2^2} \sqrt{1 - y_2^2} \\ z = z_2 + x_2 y_2 \end{cases} \end{aligned}$$

while their relations to the Weyl coordinates given just above follow from the definition of the function $\lambda(\xi, \eta, w)$.

Physical parameters of the solution

The calculations described above lead to rather complicate form of the solution expressed in terms of functions of bipolar coordinates x_1, x_2, y_1, y_2 , the formal mass and charge parameters $\tilde{m}_1, \tilde{m}_2, \tilde{e}_1, \tilde{e}_2$ and the parameters z_1 and z_2 which characterize the location of the sources on the symmetry axis. However, it is more convenient to describe the structure of this solution using certain combinations of these parameters, which, as it will be shown later, represent physical parameters – the masses of the sources m_1, m_2 , their charges e_1, e_2 calculated using the Gauss theorem and therefore, expressed in terms of the Komar-like integrals. These physical parameters are determined by the expressions

$$\begin{aligned} m_1 &= \tilde{m}_1 + \frac{\tilde{e}_1 \tilde{e}_2 (\ell - \tilde{m}_1 + \tilde{m}_2)}{P_1 P_2 + \tilde{e}_1 \tilde{e}_2} & e_1 &= \frac{\tilde{e}_1 P_2 (\ell - \tilde{m}_1 + \tilde{m}_2)}{P_1 P_2 + \tilde{e}_1 \tilde{e}_2} \\ m_2 &= \tilde{m}_2 - \frac{\tilde{e}_1 \tilde{e}_2 (\ell - \tilde{m}_1 + \tilde{m}_2)}{P_1 P_2 + \tilde{e}_1 \tilde{e}_2} & e_2 &= \frac{\tilde{e}_1 P_1 + \tilde{e}_2 P_2}{\ell - \tilde{m}_1 + \tilde{m}_2} - e_1 \end{aligned}$$

where P_1, P_2 were introduced in (29). The inverse relations are (γ is defined below):

$$\begin{aligned} \tilde{m}_1 &= \frac{(\ell + m_1 + m_2)^2 - K_0}{2(\ell + m_1 + m_2)} & \tilde{m}_2 &= \frac{(m_1 + m_2)^2 - \ell^2 + K_0}{2(\ell + m_1 + m_2)} \\ \tilde{e}_1^2 &= \frac{1}{2} [(\ell + m_2)^2 - m_1^2 - K_0] + e_1(e_1 + e_2) \\ \tilde{e}_2^2 &= \frac{1}{2} [(\ell - m_1)^2 - m_2^2 - K_0] + (e_2 + 2\gamma)(e_1 + e_2) + \frac{(m_1 + m_2)K_0}{\ell + m_1 + m_2} \\ P_1^2 &= \frac{1}{2} [(\ell - m_1)^2 - m_2^2 + K_0] + (e_2 + 2\gamma)(e_1 + e_2) - \frac{(m_1 + m_2)K_0}{\ell + m_1 + m_2} \\ P_2^2 &= \frac{1}{2} [(\ell + m_2)^2 - m_1^2 + K_0] + e_1(e_1 + e_2) \\ \tilde{e}_1 \tilde{e}_2 &= -P_1 P_2 \left(\frac{(\ell + m_2)^2 - m_1^2 - K_0}{(\ell + m_2)^2 - m_1^2 + K_0} \right) \\ K_0^2 &\equiv [(\ell + m_2)^2 - m_1^2]^2 + 4e_1(\ell + m_1 + m_2)[e_2(\ell - m_1 + m_2) + 2e_1 m_2] \end{aligned}$$

The parameters σ_1 and σ_2 also can be expressed in terms of m_1, m_2, e_1, e_2 and ℓ :

$$\begin{aligned} \sigma_1^2 &= \tilde{m}_1^2 - \tilde{e}_1^2 = m_1^2 - e_1^2 + 2e_1\gamma, \\ \sigma_2^2 &= \tilde{m}_2^2 - \tilde{e}_2^2 = m_2^2 - e_2^2 - 2e_2\gamma, \end{aligned} \quad \text{where} \quad \gamma = \frac{m_2 e_1 - m_1 e_2}{\ell + m_1 + m_2} \quad (31)$$

Using these parameters, we obtain more symmetric and simpler form of the solution. We introduce also, instead of e_1 and e_2 , two other parameters q_1 and q_2 such that:

$$\begin{aligned} q_1 &= e_1 - \gamma, & q_2 &= e_2 + \gamma, & \gamma &= \frac{m_2 q_1 - m_1 q_2}{\ell}, & \sigma_1^2 &= m_1^2 + \gamma^2 - q_1^2, \\ & & & & & & \sigma_2^2 &= m_2^2 + \gamma^2 - q_2^2. \end{aligned} \quad (32)$$

Finally, after some long calculations and consideration of various possible forms of our solution, its rather short form was found. In this form, all components of the solution are presented as functions of six parameters with only one constraint

$$\{m_1, m_2, q_1, q_2, \ell, \gamma\} \quad \left\| \quad \gamma\ell = m_2 q_1 - m_1 q_2 \quad (33)\right.$$

which leaves only five parameters to be independent.

5-parametric solution for interacting Reissner-Nordström sources

For the monodromy data (20),(21), the corresponding solution is static and its metric and electromagnetic potential in cylindrical Weyl coordinates take the forms

$$\begin{aligned} ds^2 &= H dt^2 - f(d\rho^2 + dz^2) - \frac{\rho^2}{H} d\varphi^2, \\ A_t &= \Phi, \quad A_\rho = A_z = A_\varphi = 0, \end{aligned} \quad (34)$$

where H, f and Φ are real functions of ρ and z . The calculations described in previous subsections lead to the following structure of the functions H, Φ and f :

$$H = \frac{\mathcal{D}^2 - \mathcal{G}^2 + \mathcal{F}^2}{(\mathcal{D} + \mathcal{G})^2}, \quad \Phi = \frac{\mathcal{F}}{\mathcal{D} + \mathcal{G}}, \quad f = \frac{f_0(\mathcal{D} + \mathcal{G})^2}{(x_1^2 - \sigma_1^2 y_1^2)(x_2^2 - \sigma_2^2 y_2^2)} \quad (35)$$

where $\mathcal{D}, \mathcal{G}, \mathcal{F}$ are polynomial functions of bipolar coordinates with rather simple coefficients depending on the parameters of the solution:

$$\begin{aligned} \mathcal{D} &= x_1 x_2 - \gamma^2 y_1 y_2 \\ &\quad + \delta [x_1^2 + x_2^2 - \sigma_1^2 y_1^2 - \sigma_2^2 y_2^2 + 2(m_1 m_2 - q_1 q_2) y_1 y_2] \\ \mathcal{G} &= m_1 x_2 + m_2 x_1 + \gamma(q_1 y_1 + q_2 y_2) \\ &\quad + 2\delta [m_1 x_1 + m_2 x_2 + y_1(q_2 \gamma - m_1 \ell) + y_2(q_1 \gamma + m_2 \ell)] \\ \mathcal{F} &= q_1 x_2 + q_2 x_1 + \gamma(m_1 y_1 + m_2 y_2) \\ &\quad + 2\delta [q_1 x_1 + q_2 x_2 + y_1(m_2 \gamma - q_1 \ell) + y_2(m_1 \gamma + q_2 \ell)] \end{aligned} \quad (36)$$

In (35), f_0 is an arbitrary constant which should be chosen so that $f \rightarrow 1$ at spatial infinity and the parameter δ in (36) is determined by the expression:

$$f_0 = \frac{1}{(1 + 2\delta)^2}, \quad \delta = \frac{m_1 m_2 - q_1 q_2}{\ell^2 - m_1^2 - m_2^2 + q_1^2 + q_2^2} \quad (37)$$

In accordance with (32) and (33), these equations give us an explicit expression of the solution in terms of five free real parameters m_1 , m_2 , e_1 , e_2 and $\ell = z_2 - z_1$. We consider this solution as depending on six parameters m_1 , m_2 , q_1 , q_2 , ℓ and γ restricted by the only one constraint (33) and with the expressions (32) for σ_1^2 , σ_2^2 :

$$\gamma\ell = m_2q_1 - m_1q_2, \quad \sigma_1^2 = m_1^2 + \gamma^2 - q_1^2, \quad \sigma_2^2 = m_2^2 + \gamma^2 - q_2^2. \quad (38)$$

This solution is asymptotically flat. As it will be explained in the next section, the parameters m_1 and m_2 characterize the individual masses of the sources and the charges of these sources are $e_1 = q_1 + \gamma$ and $e_2 = q_2 - \gamma$ respectively, while the total mass $m = m_1 + m_2$ and the total charge $e = e_1 + e_2$.

Physical parameters of the sources

For electrovacuum space-time which admits a time-like Killing vector field $\xi = \partial_t$ we can write (following, for example,³) the "dynamical" part of the Einstein-Maxwell equations in the Kinnersley-like self-dual form

$$\nabla_k^+ \mathcal{H}^{ik} = 0, \quad \nabla_k^+ \mathcal{F}^{ik} = 0 \quad (39)$$

where the bivectors \mathcal{H}_{ik} and \mathcal{F}_{ik} are defined as follows ($\gamma = c = 1$):

$$\begin{aligned} \mathcal{H}_{ik}^+ &\equiv \mathcal{K}_{ik}^+ - 2\bar{\Phi}\mathcal{F}_{kl}^+, \quad \mathcal{K}_{ik}^+ = K_{ik} + \frac{i}{2}\varepsilon_{iklm}K^{lm}, \quad K_{ik} = \partial_i\xi_k - \partial_k\xi_i, \\ \mathcal{F}_{ik}^+ &= F_{ik} + \frac{i}{2}\varepsilon_{iklm}F^{lm}, \quad F_{ik} = \partial_iA_k - \partial_kA_i \end{aligned} \quad (40)$$

and, due to (39), these self-dual bivectors possess complex vector potentials, one of which determines the complex scalar function Φ which enters the expressions (40):

$$\mathcal{H}_{ik}^+ = \partial_i\mathcal{H}_k - \partial_k\mathcal{H}_i, \quad \mathcal{F}_{ik}^+ = \partial_i\Phi_k - \partial_k\Phi_i, \quad \Phi \equiv \xi^k\Phi_k \quad (41)$$

The "dynamical" equations (39) allow us to construct the "conserved" quantities – the additive integral values which characterize the sources and which can be calculated as the integrals over the spherical-like 2-surfaces surrounding different parts of the sources on the space-like hypersurfaces slicing the space-time region outside the sources. For stationary axisymmetric spacetime outside the field sources we consider the space-like hypersurfaces Σ_t : $t = \text{const}$ where t is the Killing parameter. Contraction the equations (39) with the gradient $\partial_i t$, we obtain

$$\overset{(3)}{\nabla}_\delta(N_i\mathcal{H}^{i\delta}) = 0, \quad \overset{(3)}{\nabla}_\delta(N_i\mathcal{F}^{i\delta}) = 0 \quad (42)$$

where $\delta = 1, 2, 3$; $\overset{(3)}{\nabla}$ is a covariant derivative with respect to a three-dimensional metric on the hypersurfaces Σ_t and N_i is a unit time-like future-directed normal to these hypersurfaces. Integrating the above equations on Σ_t over a three-dimensional region between a sphere of a large radius B_∞ located in the asymptotically Minkowski region and by a closed surface B_s surrounding the sources, and

applying the Gauss theorem, we obtain

$$\int_{B_s} (\mathcal{H}_{ik}^+ N^i n^k) d^2\sigma = \int_{B_\infty} (\mathcal{H}_{ik}^+ N^i n^k) d^2\sigma, \quad \int_{B_s} (\mathcal{F}_{ik}^+ N^i n^k) d^2\sigma = \int_{B_\infty} (\mathcal{F}_{ik}^+ N^i n^k) d^2\sigma \quad (43)$$

where n_k is a unit vector tangent to the hypersurface Σ_t and representing the outward normal to the corresponding boundary B_s or B_∞ and $d^2\sigma$ is the area element on these boundaries. At first, we calculate the integrals over B_∞ .

For any stationary axisymmetric asymptotically flat electrovacuum solution of Einstein - Maxwell equations, the metric and electromagnetic potential of the form (4) at spatial infinity admit the expansions (the NUT parameter, the total magnetic charge and the additive constants in A_t and A_φ are assumed to vanish):

$$\begin{aligned} g_{tt} &= 1 - \frac{2m}{r} + O\left(\frac{1}{r^2}\right) & f &= 1 + \frac{2m}{r} + O\left(\frac{1}{r^2}\right) \\ g_{t\varphi} &= \frac{2am}{r} \sin^2 \theta + O\left(\frac{1}{r^2}\right) & A_t &= \frac{e}{r} + O\left(\frac{1}{r}\right) \\ g_{\varphi\varphi} &= -r^2 \sin^2 \theta - 2mr \sin^2 \theta + O(r^0) & A_\varphi &= O\left(\frac{1}{r}\right) \end{aligned}$$

where $\rho = r \sin \theta$, $z = z_* + r \cos \theta$ and the constants m , a and e mean the total mass, total angular momentum per unit mass and the total electric charge respectively. For the complex vector potentials \mathcal{H}_i and Φ_i introduced in (41) for $r \rightarrow \infty$ in the coordinates $\{t, \rho, z, \varphi\}$ we obtain the expansions

$$\begin{aligned} \mathcal{H}_i &= \left\{ -\frac{2m}{r} + O\left(\frac{1}{r^2}\right), 0, 0, -2im \cos \theta + O\left(\frac{1}{r}\right) \right\} \\ \Phi_i &= \left\{ \frac{e}{r} + O\left(\frac{1}{r^2}\right), 0, 0, ie \cos \theta + O\left(\frac{1}{r}\right) \right\} \end{aligned} \quad (44)$$

The components of the vector N^i in these coordinates take the form

$$N^i = \left\{ 1 + \frac{m}{r} + O\left(\frac{1}{r^2}\right), 0, 0, \frac{2am}{r^3} + O\left(\frac{1}{r^4}\right) \right\}$$

For a sphere of a large radius, the spatial unit normal vector in the leading order is $n^i \partial_i = \partial / \partial r$ and in the limit $r \rightarrow \infty$ for the integrals (43) over B_∞ we obtain

$$\int_{B_\infty} (\mathcal{H}_{ik}^+ N^i n^k) d^2\sigma = -8\pi m, \quad \int_{B_\infty} (\mathcal{F}_{ik}^+ N^i n^k) d^2\sigma = 4\pi e$$

where m and e are the total mass and charge of the field configuration. This allows us to express (in accordance with (43)) the total mass and charge of a system of sources in terms of the integrals over the surface B_s surrounding the sources:

$$m = -\frac{1}{8\pi} \int_{B_s} (\mathcal{H}_{ik}^+ N^i n^k) d^2\sigma, \quad e = \frac{1}{4\pi} \int_{B_s} (\mathcal{F}_{ik}^+ N^i n^k) d^2\sigma \quad (45)$$

Now we transform the integrands in (45) using the self-duality of \mathcal{H}^+ and \mathcal{F}^+ :

$$\left. \begin{aligned} \int_{B_s} \mathcal{H}^{ij} N_i n_j d^2 \sigma &= i \int_{B_s} \mathcal{H}_{ij} k^i l^j d^2 \sigma \\ \int_{B_s} \mathcal{F}^{ij} N_i n_j d^2 \sigma &= i \int_{B_s} \mathcal{F}_{ij} k^i l^j d^2 \sigma \end{aligned} \right\} \left\{ \begin{aligned} \varepsilon^{ijkl} N_k n_l &= k^i l^j - k^j l^i \\ \varepsilon^{ijkl} N_k n_l k_i l_j &= 1 \end{aligned} \right. \quad (46)$$

Here we introduced two spatial unit vectors k^i and l^i which are orthogonal to each other and tangent to 2-surface B_s . These vectors determine the orientation of the 2-surface B_s so that the spatial basis $\{n^i, k^j, l^k\}$ is positively oriented. Choosing the 2-surface B_s to be axially symmetric, we specify the choice of the vectors k^i and l^j in the tangent space of B_s so that l^j would be the rotational Killing vector $\partial/\partial\varphi$ and k^i is tangent to a curve \mathcal{L} on B_s with $\varphi = \text{const}$:

$$k^i = \{0, \frac{d\rho}{d\ell}, \frac{dz}{d\ell}, 0\} \quad l^i = \frac{1}{\sqrt{-g_{\varphi\varphi}}} \{0, 0, 0, 1\}$$

where the parameter ℓ is the length on the curve \mathcal{L} and the direction on \mathcal{L} is chosen so that it goes from some point on the positive part of the z -axis to some point on its negative part (provided the point $\rho = z = 0$ is located somewhere inside B_s). The element of the area on B_s is $\sqrt{-g_{\varphi\varphi}} d\ell d\varphi$. After the integration over φ we observe that the integrands of the remaining contour integrals over \mathcal{L} are the differentials of the φ -components of the potentials of self-dual bivectors \mathcal{H}^+ and \mathcal{F}^+ :

$$\begin{aligned} m &= -\frac{i}{4} \int_{\mathcal{L}} (\partial_\mu \mathcal{H}_\varphi) dx^\mu = \frac{i}{4} [(\mathcal{H}_\varphi)_+ - (\mathcal{H}_\varphi)_-] \\ e &= \frac{i}{2} \int_{\mathcal{B}_f \mathcal{L}} (\partial_\mu \Phi_\varphi) dx^\mu = -\frac{i}{2} [(\Phi_\varphi)_+ - (\Phi_\varphi)_-] \end{aligned} \quad (47)$$

where $(\dots)_+$ means the value of a potential at the beginning of \mathcal{L} , i.e. on the positive part of the z -axis, and $(\dots)_-$ – its value on the negative part of this axis.

It is important to note that \mathcal{H}_φ and Φ_φ are constant along the regular parts of the symmetry axis outside the sources and therefore, the integrals (47) do not change if we deform the surface of integration B_s surrounding the sources in the domain of regularity of the solution.

Now we consider the case, in which the source of the field consists of two parts (black holes, naked singularities or extended bodies) separated by some segment of the symmetry axis. In this case, we can deform the surface B_s surrounding the sources into two spheres each surrounding one of these parts of the sources and a thin tube surrounding the intermediate part of the axis between the sources. If \mathcal{H}_φ and Φ_φ are constant on the intermediate part of the axis, the integral over the thin tube vanish and we obtain that the integrals for the total mass and the total charge are expressed as the sums of the integrals of the same type over the

spheres surrounding each of the sources. In particular, for the sources consisting of two parts, as it is in our solution for two Reissner - Nordström sources, we obtain

$$m = m_1 + m_2, \quad e = e_1 + e_2$$

where m_1 , m_2 and e_1 , e_2 can be interpreted respectively as the masses (energies) and charges of the corresponding parts of the source. To conclude this section, we mention that rather tedious calculations show that the parameters m_1 , m_2 and e_1 , e_2 in our electrostatic solution presented in the previous section and describing the field of two Reissner - Nordström sources, coincide with the values of the integrals (47) calculated for each part of the sources and therefore, these parameters can be interpreted as the additive masses and charges of these interacting sources.

Equilibrium of two Reissner - Nordström sources

The 5-parametric solution (34) - (38) presented above is asymptotically flat and the space-time geometry is regular far enough from the sources. However, for an arbitrary choice of parameters m_1 , m_2 , q_1 , q_2 and ℓ , this solution may have some physically not reasonable singularities ("struts") on the part of the axis of symmetry between the sources. Like at the vertex of the cone, at these points, the local Euclidean properties of space-time may be violated so that on the space-time sections $t = \text{const}$, $z = \text{const}$ the ratio of the length \mathcal{L} of a small circle, surrounding the axis of symmetry $\rho = 0$ and contracting to the point z of this axis, to its radius \mathcal{R} multiplied by 2π is not equal to a unit. The constraint imposed on the parameters which provide the absence of such conical singularities plays the role of equilibrium condition because it allows to select a physically acceptable solution in which these singularities are absent and the sources are in the equilibrium because of the balance between their gravitational and electromagnetic interactions.

As it follows from elementary considerations, the limiting value of the square of the ratio $(2\pi\mathcal{R})/\mathcal{L}$ for $\mathcal{R} \rightarrow 0$ is equal to the value of the product fH at the point z of the axis. However, the field equations imply that on the regular parts of the axis of symmetry the product $f(\rho = 0, z)H(\rho = 0, z)$ does not depend on z and therefore, it is a constant. However, the values of this constant can be different on different parts of the axis of symmetry separated by the sources, and the condition $fH = 1$ of the absence of conical singularities should be satisfied at each of these parts of the axis of symmetry.

In our solution with two separated sources, there exists three parts of the axis of symmetry where the condition of the absence of conical points should be considered. There are two semi-infinite parts of the axis outside the sources. On the negative part L_- , we have $-\infty < z < z_1 - \sigma_1$ (if the first source is a black hole) or $-\infty < z < z_1$ (if the first source is a naked singularity) and on this part $x_1 = z_1 - z$, $x_2 = z_2 - z$, $y_1 = -1$, $y_2 = -1$. On the positive part L_+ of the axis, $z_2 + \sigma_2 < z < \infty$ (if the second source is a black hole) or $z_2 < z < \infty$ (if the second source is a naked singularity) and we have there $x_1 = z - z_1$, $x_2 = z - z_2$, $y_1 = 1$, $y_2 = 1$. It is easy to

see that on these parts of the axis the condition $fH = 1$ is satisfied for any choice of free parameters of our solution. However, it is not the case on the part L_0 of the axis between the sources. On L_0 we have $z_1 + Re(\sigma_1) < z < z_2 - Re(\sigma_2)$ and $x_1 = z - z_1$, $x_2 = z_1 + \ell - z$, $y_1 = 1$, $y_2 = -1$ and the product fH also takes there some constant value whose equality to a unit give us the equilibrium condition:

$$m_1 m_2 - q_1 q_2 = 0 \quad (48)$$

It is interesting to note that this equilibrium condition looks just like the Newtonian condition of equilibrium of two charged point-like masses, but in the case of General Relativity this condition relates the masses of the Reissner-Nordström sources not with their charges e_1 and e_2 , but with the parameters q_1 and q_2 whose expressions in terms of masses m_1 , m_2 and charges e_1 , e_2 depend also on the z -distance ℓ separating the sources.

The equilibrium condition (48) allows also to simplify the solution (34) - (38) and leads to the 4-parametric family which describes the superposed field of two Reissner - Nordström sources in equilibrium. This solution has been presented in our previous short paper.¹ Here we present it in a bit different form using m_1 , m_2 , q_1 , q_2 , γ as the basic set of parameters, such that the first four of them should satisfy the equilibrium condition (48). For this solution the metric functions have the same expressions (35) where the polynomials \mathcal{D} , \mathcal{G} and \mathcal{F} have the expressions

$$\left. \begin{aligned} \mathcal{D} &= x_1 x_2 - \gamma^2 y_1 y_2 \\ \mathcal{G} &= m_1 x_2 + m_2 x_1 + \gamma(q_1 y_1 + q_2 y_2) \\ \mathcal{F} &= q_1 x_2 + q_2 x_1 + \gamma(m_1 y_1 + m_2 y_2) \end{aligned} \right\} \quad m_1 m_2 = q_1 q_2$$

and $f_0 = 1$. The other physical parameters – the z -distance ℓ separating the sources and their charges are determined by the expressions

$$\ell \equiv z_2 - z_1 = (m_2 q_1 - m_1 q_2)/\gamma, \quad e_1 = q_1 + \gamma, \quad e_2 = q_2 - \gamma$$

The independent parameters m_1 , m_2 , q_1 , q_2 , γ of this solution should be chosen so that the sources would be separated actually by some positive distance, i.e. $\ell > Re(\sigma_1) + Re(\sigma_2)$.

This concludes our description in this paper of some fragments of the monodromy transform approach and of the procedure for solution of the corresponding singular integral equations for stationary axisymmetric electrovacuum fields with simple rational monodromy data, which leads to the construction of the 5-parametric family of solutions for the field of two interacting Reissner-Nordström sources, and of the derivation from this solution of the 4-parametric family of solutions for the fields of equilibrium configurations of these sources. Some properties of these equilibrium field configurations have been discussed in,¹ however, a more detail analysis of physical and geometrical properties of these configurations, such as the structure of the superposed fields, influence on the geometry of horizons and on the space-time geometry inside the horizon of the external gravitational and electromagnetic fields

created by another source, tidal influence of these fields on the structure of naked singularities, stability of equilibrium and probably, some others, are expected to be the subject of our next publications.

Acknowledgements

The authors are thankful to the Organizing Committee of MG11 for the invitation to submit this paper devoted to mathematical aspects of solving of Einstein-Maxwell equations to the Proceedings of this Meeting.

GAA is thankful to ICRAnet for the financial support and hospitality during his visit to ICRAnet (Pescara, Italy) during June 2007, when this paper was started. The work of GAA was also supported in parts by the Russian Foundation for Basic Research (grants 05-01-00219, 05-01-00498, 06-01-92057-CE) and the programs "Mathematical Methods of Nonlinear Dynamics" of the Russian Academy of Sciences, and "Leading Scientific Schools" of Russian Federation (grant NSh-4710.2006.1).

References

1. G.A. Alekseev and V.A. Belinski, Phys. Rev. **D 76**, 021501(R) (2007).
2. G.A. Alekseev, *Sov. Phys. Dokl.* **30**, 565 (1985); Theoretical and Mathematical Physics, 143(2), 720–740 (2005); in Proceedings of 11th Marcel Grossmann Meeting on General Relativity, Berlin (2007).
3. G.A. Alekseev, Proc. Steklov Math. Inst., Providence, RI: American Math. Soc., **3**, 215 (1988); Abstracts of Contributed Papers, 13th International Conference on General Relativity and gravitation, edited by Pedro W. Lamberty, Omar E. Ortiz (Huerta Grande, Cordoba, Argentina), p. 3 - 4 (1992).
4. V.A. Belinski and V.E. Zakharov, Sov. Phys. JETP, **48**, 985 (1978).
5. W. Kinnersley, J. Math. Phys. **18**, 1529 (1977); W. Kinnersley and D. M. Chitre, J. Math. Phys. **18**, 1538 (1977).
6. I. Hauser and F. J. Ernst, Phys. Rev. **D20**, 362 (1979); Phys. Rev. **D20**, 1783 (1979).
7. N.R. Sibgatullin, *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields*, Nauka, Moscow (1984); English translation: Springer-Verlag, Berlin (1991).
8. G.A. Alekseev and A.A. Garcia, Phys. Rev. **D53**, 1853 (1996).
9. V. Belinski and E. Verdaguer, *Gravitational Solitons*, CUP, Cambridge (2001).
10. G.A. Alekseev, Proceedings of GR9, **1**, 2 (1980); JETP Lett. **32** 277 (1980).